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An Optimality Criterion Method for Large-Scale Structures

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An optimality criterion method, which exploits the concept of one most critical constraint, is reported. The method eliminates the need to calculate a large set of Lagrange multipliers for the active constraints, and also eliminates the need for a decision as to whether or not a particular constraint should be considered active. The method can treat multiple load conditions and stress and displacement constraints. Application of the method to a number of truss and frame structures demonstrates the efficiency and accuracy of the method.

I. Introduction

THE problem of structural optimization has become of great interest to many researchers during the past few years. The primary goal of this recent work has been to obtain a minimum weight structure subject to various constraints in minimal computational time and with minimal computer storage. The efficiency of earlier painfully slow mathematical programming techniques for large structural problems has been improved considerably by Schmit, Farshi, and Muira.[1-3] Physical optimality criterion techniques to efficiently design large-scale structures have been developed.[4-7] In addition, Dobbs and Nelson[8] and Rizzi[9] have recently used mathematical optimality criterion methods based on the Kuhn-Tucker conditions to obtain minimum weight designs efficiently. Khan et al.[10] applied efficient physical optimality criterion techniques to simple structures and complex high-speed mechanisms.

The development of the method presented here was motivated by a desire to extend to problems with multiple constraints of different types (i.e., stress and displacement constraints) the simplicity inherent in physical optimality criterion methods developed for single constraints of each type. For instance, the stress ratio method has, over the years, demonstrated a remarkable ability to efficiently produce minimum weight designs or near-minimum weight designs for a great variety of multiloaded structures under stress constraints. Likewise, physical optimality criterion methods for displacement constraints have been derived and applied with success.

Each of these independent physical optimality criterion methods gives rise to a simple recursion formula for redesign. If there is only one type of constraint (i.e., either stress or displacement or buckling), the redesign process requires only an analysis of the structure and an application of the appropriate recursion formula. There is no requirement, in addition to an analysis of the structure, to solve 1) a set of linear algebraic equations for a set of Lagrange multipliers (as in Ref. 8), or 2) a linear program based on a linearization of an assumed set of active and potentially active constraints (as in Ref. 1), or 3) a nonlinear programming problem[3] in the active and potentially active constraints in order to obtain a new design.

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In this paper, recursion formulas for stress and displacement constraints, which result from the Kuhn-Tucker necessary conditions for each type of constraint, are incorporated into a design algorithm which exploits the concept of a single, most critical displacement constraint. The algorithm requires only one analysis of the structure per design cycle. Redesign of each member is achieved by means of one of two recursion formulas. No sets of Lagrange multipliers need be calculated, no subsidiary LP or NLP must be solved, no decision as to active or potentially active constraints must be made, and no move limits need be used. The method is applicable to two- and three-dimensional trusses and two-dimensional frames of fixed geometry under multiple load conditions and stress and displacement constraints.

II. Theory

The design problem to be solved here can be stated as: Find the vector of design variables $A=(A_1,A_2,\ldots A_N)$ such that the volume of the structure

$$V=\sum_{i=1}^{N} A_i\ell_i \rightarrow \min \tag{1}$$

while

$$\begin{aligned} \sigma_{ik} &\le \bar{\sigma}_i \qquad i=1,\ldots,N \\ & \qquad\qquad k=1,\ldots,K \\ u_{jk} &\le \bar{u}_j \quad\; j=1,\ldots,J \end{aligned} \tag{2}$$

where A_i and ℓ_i are the cross-sectional area and length of the ith member, N is the number of members, σ_{ik} is the stress in the ith member of the kth load condition, K is the number of load conditions, and $\bar{\sigma}_i$ is the limiting stress in the ith member. Also, u_{jk} is the displacement in the jth constrained degree of freedom, $\bar{u}_j$ is the limiting value of the displacement in the jth constrained degree of freedom, and J is the number of displacement constrained degrees of freedom.

A. Stress Constraints

Considering stress constraints alone, the Kuhn-Tucker conditions for the design problem of Eqs. (1) and (2) results in the well-known stress ratio formula for redesign (see for example, Ref. 10)

$$[A_i]_{\nu+1}=\left[\left(\frac{\max_k |\sigma_{ik}|}{\bar{\sigma}_i}\right)A_i\right]_\nu \tag{3}$$

where ν is the iteration counter. If design-variable linking is used to form groups of design variables, where members of one group are the same size, Eq. (3) is applied to each member of a group and the largest A_i from Eq. (3) is taken as the size for all members of the group.

B. Displacement Constraints

Considering displacement constraints alone, the Lagrangian for the design problem of Eqs. (1) and (2) is

$$L = V + \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} (u_{jk} - \bar{u}_j) \tag{4}$$

and the Kuhn-Tucker necessary conditions for a minimum are

$$\frac{\partial V}{\partial A_i} + \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} \frac{\partial u_{jk}}{\partial A_i} = 0 \quad i = 1,\ldots,N \tag{5a}$$

$$u_{jk} - \bar{u}_j \leq 0 \text{ or } \lambda_{jk} \geq 0 \qquad j = 1,\ldots,J \qquad k = 1,\ldots,K \tag{5b}$$

Suppose now that the pth constrained displacement in the qth load condition is exactly active, and the other constrained displacements are not. The Eqs. (5) become

$$\partial V / \partial A_i + \lambda_{pq} \partial u_{pq} / \partial A_i = 0 \quad i = 1,\ldots,N \tag{6a}$$

$$u_{pq} - \bar{u}_p = 0 \quad \lambda_{pq} > 0 \tag{6b}$$

By means of the unit load theorem of structural analysis, the derivative in Eq. (6a) can be written[11]:

$$\frac{\partial u_{pq}}{\partial A_i} = -\frac{x_i^T K_i \bar{x}_i}{A_i} \tag{7}$$

where K_i is the stiffness matrix of the ith member, x_i is the displacement vector for the ith member due to the qth load condition, and $\bar{x}_i$ is the displacement vector for the ith member due to a unit load applied at the location and in the direction of the pth constrained degree of freedom. Substituting Eqs. (1) and (7) into Eqs. (6) gives

$$A_i \ell_i - \lambda_{pq} x_i^T K_i \bar{x}_i = 0 \qquad i = 1,\ldots,N \tag{8}$$

and summing Eq. (8) over all members results in

$$\lambda_{pq} = V / u_{pq} = V / \bar{u}_p \tag{9}$$

Combining Eqs. (8) and (9) results in

$$1 = (V/\bar{u}_p)[(x_i^T K_i \bar{x}_i)/A_i \ell_i] \tag{10}$$

which is the optimality criterion that must be satisfied at the optimum design. From Eq. (10), the following recursive formula results:

$$\{A_i\}_{\nu+1} = \left\{ \left[\left(\frac{V}{\bar{u}_p} \right) \left(\frac{x_i^T K_i \bar{x}_i}{A_i \ell_i} \right) \right]^{\eta} A_i \right\}_{\nu} \tag{11}$$

If design-variable linking is used and n members are to have the same design variable A_i, Eq. (11) is written as

$$\{A_i\}_{\nu+1} = \left\{ \left[\left(\frac{V}{\bar{u}_p} \right) \left(\frac{\sum_{j=1}^{n} x_j^T K_j \bar{x}_j}{A_i \sum_{j=1}^{n} \ell_j} \right) \right]^{\eta} A_i \right\}_{\nu} \tag{12}$$

where the summation over j in Eq. (12) is over those members which have common design variable A_i.

In Eqs. (11) and (12), η is a relaxation parameter used to control the rate of convergence and stability of the method. It is the only arbitrary parameter involved in the algorithm. Values between 0.001 and 0.2 have been found to be appropriate.

The derivation just presented is incorporated into the design algorithm of Sec. III. Its use is justified for multidisplacement constrained problems because of the selection of the most active (or most violated) constraint. In this method, as well as most currently available techniques, there is normally only one most active constraint at any iteration. There may be many constraints which are nearly active—this, of course, is especially true at the optimal design—but only one which is most active. The true optimal design may be one having several active constraints, but this is almost never exactly obtained in a numerical algorithm, because of finite arithmetic and because practical convergence criteria stop the process before the active constraints reach the point where they are active to the number of digits carried by the machine performing the computations. In the special case where two or more displacement constraints are exactly equal because of symmetry or other structural limitations, these exactly equal displacements are treated as one constraint. In the method presented here, this most active (or most violated) constraint in some load condition is considered to be the only active constraint; all other displacement constraints are considered inactive.

The recursion relations of Eqs. (11) and (12) have been applied to several displacement-constrained problems, but practical problems will be those with both stress and displacement constraints. Thus, the stress recursion formula of Eq. (3) has been combined with the displacement recursion formulas of Eqs. (11) and (12) and an important scaling procedure to produce a design procedure which is applicable to stress and displacement-constrained trusses and frames under multiple load conditions.

III. Design Algorithm

1) Choose any uniform design A_i, $i = 1,2,\ldots,N$. Choose a value of the relaxation parameter (η) [say, 0.15-0.08].

2) Analyze the design for each load condition.

3) Check displacments in each load condition at those nodes where displacement limitations are imposed, and determine the node and direction for which the calculated displacement most closely approaches (or exceeds) the allowable displacement. This is the most critical displacement (u_{pq}).

4) Knowing the magnitude of the most critical displacement (u_{pq}) from step 3 and the value of the allowable displacement ($\bar{u}_p$), scale the chosen design so that the most critical displacement becomes active. All other displacement constraints will then be inactive. Let the scaled design be denoted by A_i', where

$$A_i' = (|u_{pq}| / |\bar{u}_p|) A_i \qquad i = 1,2,\ldots,N \tag{13}$$

If the structure was analyzed with the scaled design, then displacement vectors calculated at step 2 would have been

$$x_i' = (|\bar{u}_p| / |u_{pq}|) x_i \qquad i = 1,2,\ldots,N \tag{14}$$

and stiffness matrix from the scaled design would be

$$K' = (|u_{pq}| / |\bar{u}_p|) K \tag{15}$$

5) From the scaled displacement vectors (x_i') and design (A_i') compute the maximum stress $\max_k |\sigma_{ik}|$ in each member i. Also, determine the stress response ratio for each member and let the most critical response ratio be obtained for the nth member. This is denoted by R_n. If $R_n > 1$, compute

$$V_I = R_n \left(\sum_{i=I}^{N} A_i' \ell_i \right)$$

or, if $R_n \leq 1$, compute

$$V_I = \sum_{i=I}^{N} A_i' \ell_i$$

6) Using the scaled design, apply a unit load only at the node and in the direction of the active displacement constraint. Let the set of resulting nodal displacements be denoted by $\tilde{x}_i'$.

Note that this is the *only* unit load that needs to be applied, and that the structural stiffness matrix inverted at step 2 is used here as scaled in step 4 to compute $\tilde{x}_i'$.

7) From Eq. (7), compute

$$\partial u_{pq} / \partial A_i' = -(x_i'^T K_i' \tilde{x}_i') / A_i' \tag{16}$$

Also, the Lagrange multiplier associated with the critical displacement is computed from Eq. (9) as

$$\lambda_{pq} = \sum_{i=I}^{N} \frac{A_i' \ell_i}{\bar{u}_p} \tag{17}$$

8) Group the member as follows: a) If $\partial u_{pq} / \partial A_i' \leq 0$ or $\sigma_i \geq \bar{\sigma}_i$, member i belongs to group G_I; b) otherwise, member i belongs to group G_2. Note that either group could be empty and a particular member would belong to only one group at a time.

9) Use the stress ratio formula, Eq. (3), to resize the elements of G_I as

$$[A_i]_{\nu+I} = \left[\left(\frac{\max\limits_k |\sigma_{ik}|}{\bar{\sigma}_i} \right) A_i' \right]_\nu$$

10) Resize the elements of G_2 using Eq. (11) [or Eq. (12)] as

$$(A_i)_{\nu+I} = \left\{ \left[\lambda_{pq} \frac{\partial u_{pq} / \partial A_i'}{\ell_i} \right]^\eta (A_i') \right\}_\nu \tag{18}$$

11) Scale the design and compute the new critical response ratio R_n' and new V_I' using steps 2-5. If the quantity $|(V_I - V_I')/V_I'|$ is less than ϵ (a small number ranging between 0.001-0.010), then go to step 12; otherwise, check the following: a) If $V_I' < V_I$, continue with step 6 with the old value of η; or b) if $V_I' > V_I$, $R_n < 1$, and $R_n' > 1$, the designer may stop at this point and the design of the previous iteration would be very close to the optimum design. Otherwise, η is reduced to one-third or one-quarter of the starting value and the process is continued with step 6.

12) If the converged design of step 5 is completely displacement-dominated, then R_n' would be less than 1 and this design is the optimal design. If the converged design of step 5 is completely stress-dominated, that is, all members are in G_I and hence overstressed, simply scale the design (multiplying all the design variables with R_n') so that no stress constraint is violated to achieve the optimum design. If the converged design has some member overstressed while others are not, then the following situations may occur: a) If $V_I < V_I'$ and $(R_n' - 1) \leq 0.05$, scale the design of the previous iteration by multiplying all the design variables with R_n; this is then taken to be optimum design. b) If $V_I' < V_I$ and $(R_n' - I) \leq 0.05$, scale the design of the current iteration by multiplying all the design variables by R_n'; this is then taken to be the optimum design. c) If $(R_n' - I) > 0.05$, reduce the value of η to one-half or one-third of the starting value and go back to step 6 and repeat the process.

The relaxation parameter (η) is the only arbitrary parameter in the design procedure. It controls the stability and convergence of this method. Experience indicates that a value of η between 0.001 to 0.2 results in optimum designs being obtained without difficulty. It is important to note that selecting the value from this range does not affect the optimum design. The same design will be obtained using any value of η between 0.001-0.2, but it will be located in fewer iterations with the larger values. One difficulty with the larger values of η is that the technique brings the design close to the optimum in a very few analyses, but oscillations will occur very close to the optimum. This is easily detected when, at a particular iteration, the scaled design weighs more than the previous design. When this occurs, η is reduced and the procedure is stabilized.

IV. Results

In this section, results for six classical truss examples and two frame examples are presented. These are intended to show the efficiency and accuracy of the design algorithm of Sec. III.

A. Ten-Bar Truss

This is a cantilever truss shown in Fig. 1, which has been studied by many researchers.[1,2,4,5,8,9] The material is aluminum of specific weight $\rho = 0.1$ lb/in.^{3} and modulus of elasticity $E = 10 \times 10^6$ psi. Displacement limits of ± 2.0 in. are imposed on all nodes in both directions, and the limiting value of stress in each member is $\pm 25{,}000$ psi. No design-variable linking is used, so there are 10 independent design variables. Two cases are considered: case 1 has $P_I = 100$ K, $P_2 = 0$; case 2 has $P_I = 150$ K, $P_2 = 50$ K. A single loading condition is considered in each case. A lower limit on member size of 0.1 in.^{2} is enforced.

The final design for case 1 is given in Table 1a. In this case, the problem was started with a uniform design with each cross-sectional area equal to 100 in.^{2}. A starting value of $\eta = 0.2$ was chosen and automatically changed to 0.05 as the design came close to the optimum. At iteration 15, a weight of 5085 lb was obtained and the design was similar to one previously reported by other researchers. However, the algorithm did not stop automatically until iteration 18, at which point the weight dropped to 5067 lb, the displacement of node 1 in the y direction was -2.0 in., the displacement of node 4 in the y direction was 0.4% below its limiting value, and member 5 had stress 2.71% below its yield value.

The final design for case 2 is given in Table 1b. This problem was started with the same initial design and η value as for case 1. The design was automatically converged at iteration 9 when member 5 had its stress equal to the limiting value, and the displacement of node 4 in the y direction was 0.3% below its specified limit. The final design obtained is in good agreement with previous designs.

B. Four-Bar Space Truss

The structure is a four-bar pyramid truss shown in Fig. 2. The material is aluminum with $\rho = 0.1$ lb/in.^{3} and $E = 10 \times 10^6$ psi. Stress limits of $\pm 25{,}000$ psi are imposed on

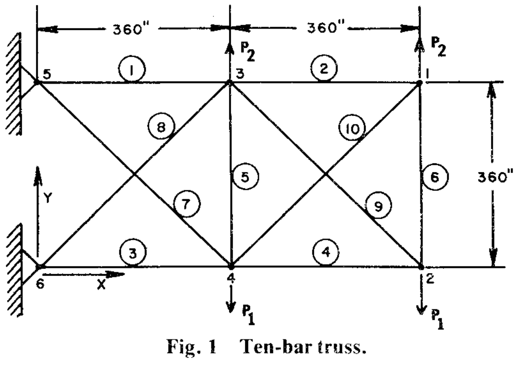

Fig. 1 Ten-bar truss.

Table 1a Comparison of final designs for ten-bar truss, case 1

Members in group no.	Final cross-sectional areas, in.^{2} Schmit and Miura NEWSUMT[2]	CONMIN[2]	Schmit and Farshi[1]	Venkayya[4]	Gellatly and Berke[5]	Dobbs and Nelson[8]	Rizzi[9]	This paper
1	30.670	30.57	33.432	30.416	31.350	30.500	30.731	30.980
2	0.100	0.369	0.100	0.128	0.100	0.100	0.10	0.10
3	23.760	23.97	24.260	23.408	20.030	23.290	23.934	24.169
4	14.590	14.73	14.260	14.905	15.600	15.428	14.733	14.805
5	0.100	0.10	0.100	0.101	0.140	0.100	0.100	0.100
6	0.100	0.364	0.100	0.101	0.240	0.210	0.100	0.406
7	8.578	8.547	8.388	8.696	8.350	7.649	8.542	7.547
8	21.070	21.11	20.740	21.084	22.210	20.980	20.954	21.046
9	20.960	20.77	19.690	21.077	22.060	21.818	21.836	20.937
10	0.100	0.320	0.100	0.186	0.100	0.100	0.100	0.100
Wt, lb	5076.85	5107.3	5089.0	5084.9	5112.0	5080.0	5076.66	5066.98
Analyses	13	14	24	26	19	15	11	18[a]

[a] A weight of 5085 lb was achieved after 15 analyses.

Table 1b Comparison of final designs for ten-bar truss, case 2

Members in group no.	Final cross-sectional areas, in.^{2} Schmit and Miura NEWSUMT[2]	CONMIN[2]	Schmit and Farshi[1]	Venkayya[4]	Gellatly and Berke[5]	Dobbs and Nelson[8]	Rizzi[9]	This paper
1	23.550	23.55	24.289	25.190	...	25.813	23.533	24.716
2	0.100	0.176	0.100	0.363	...	0.100	0.100	0.100
3	25.290	25.20	23.346	25.419	...	27.233	25.291	26.541
4	14.360	14.39	13.654	14.327	...	16.653	14.374	13.219
5	0.100	0.100	0.100	0.417	...	0.100	0.100	0.108
6	1.970	1.967	1.969	3.144	...	2.024	1.9697	4.835
7	12.390	12.400	12.670	12.083	...	12.776	12.389	12.664
8	12.810	12.860	12.544	14.612	...	14.218	12.825	13.775
9	20.340	20.410	21.971	20.261	...	22.137	20.328	18.438
10	0.100	0.100	0.100	0.513	...	0.100	0.100	0.10
Wt, lb	4676.96	4684.11	4691.84	4895.60	...	5059.7	4676.92	4792.52
Analyses	11	10	23	13	...	12	12	9

all members. No design-variable linking is used. Two cases are considered: case 1 has a loading of $P_x = 10$ K, $P_y = 20$ K, and $P_z = -60$ K, and a displacement limit of ± 0.3 in. is imposed at the top joint in the z direction; case 2 has a loading of $P_x = 40$ K, $P_y = 100$ K, $P_z = -30$ K, and displacement limits at the top joint are ± 0.3 in. in the x direction, 0.5 in the y direction, ± 0.4 in. in the z direction. Results are given in Table 2. This table shows good correspondence, with previous results, of the design obtained with the new method and its efficiency. The initial design for both cases had all members at 100 in.^{2}. In both cases 1 and 2, member 3 had stress equal to its limiting value, while in case 1 displacement of the top node in the z direction was 3.8% below its limit and in case 2 displacement in the y direction was 1.9% below.

C. Twenty-Two-Member Space Truss

This structure, shown in Fig. 3, has each joint connected to every other joint by a member, except that members between support joints are excluded. It was studied in Ref. 12 in the context of determining the global optimum of trusses with vanishing members.

All members are aluminum with $E = 10 \times 10^6$ psi and $\rho = 0.1$ lb/in.^{3}. The 22 members are linked into 7 groups as shown in Table 3. Table 3 gives the limiting stresses for each group of members. Displacement constraints of ± 0.2 in. at all nodes in all directions are imposed, and a minimum member size of 0.1 in.^{2} holds if a member is not prescribed to vanish. Three load conditions, as given in Table 3a of Ref. 12 or Table 4 of Ref. 15, are considered in each of three design cases. Case 1 has all groups of members nonvanishing; case 2 has the members of group 4 set to zero; and in case 3, the members of group 3 vanish. Table 4 summarizes the results of the three cases obtained by the method of this paper and compares them with the results of Ref. 12. Case 1 is the global optimum for this truss. The present method achieves a design with weight

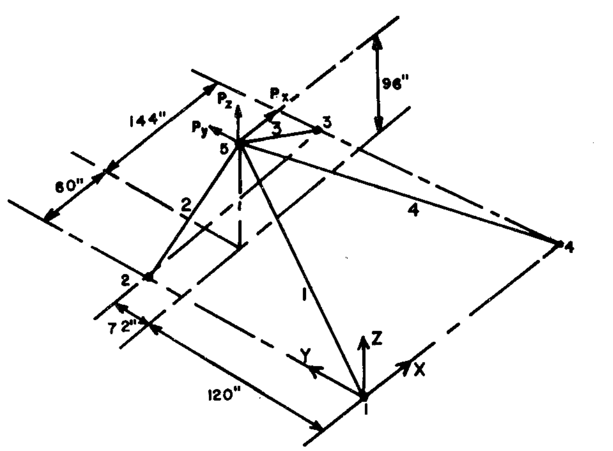

Fig. 2 Four-bar space truss.

Table 2a Final designs, four-bar pyramid, case 1

Member no.	Final, cross-sectional areas, in.^{2} Schmit and Farshi[1]	Venkayya[4]	This paper
1	0.0	0.277	0.0
2	3.765	4.1527	3.651
3	0.769	0.746	0.769
4	2.514	2.477	2.759
Wt, lb	117.89	126.43	121.50
Analyses	16	37	6

Table 2b Final designs, four-bar pyramid, case 2

Member no.	Final cross-sectional areas, in.^{2} Schmit and Farshi[1]	Venkayya[4]	This paper
1	3.210	3.147	3.419
2	2.614	2.147	2.511
3	2.159	2.162	2.159
4	0.0	0.0	0.0
Wt, lb	128.53	128.561	130.625
Analyses	14	...	7

Table 3 Member linking groups and stress limits, twenty-two-member space truss

Design variable group number	Members of group	Lower limiting stress, psi	Upper limiting stress, psi
1	1, 2, 3, 4	24,000	36,000
2	5, 6	30,000	
3	7, 8	28,000	
4	9, 10	26,000	
5	11, 12, 13, 14	22,000	
6	15, 16, 17, 18	20,000	
7	19, 20, 21, 22	18,000	36,000

within 1% of the global minimum weight in five analyses. Cases 2 and 3 converge to designs very close to the results of Ref. 12 in just six analyses.

The initial design for all three cases was uniform with all members at 10 in.^{2}. The starting values of parameter η for the three different cases were arbitrarily chosen to be 0.2, 0.125, and 0.1, respectively. These changed to one-quarter of their starting values at the end of the optimization process. Also, the design process was studied by starting all three cases with the same value of η. The final designs obtained were the same as those presented in Table 4.

D. Twenty-Five-Bar Transmission Tower Truss

This much-studied truss[1,2,4,5,8,9] is shown in Fig. 4. The materials of all members is again aluminum with $E = 10 \times 10^6$ psi and $\rho = 0.1$ lb/in.^{3}. Design-variable linking is used to reduce the number of independent design variables from 25 to 8. The member groups and the stress limits for each group are the same as those given in Table 32 of Ref. 2. Two load conditions are considered; these are given in Table 31 of Ref. 2. Displacement limits of ± 0.35 in. are imposed on every node in every direction. Table 5 gives the final design obtained and compares this with previously obtained designs. The comparison indicates that the method of this paper gave a design similar to those previously obtained but with a weight about 2% higher. The problem was started with η equal to 0.1 and all members at 100 in.^{2}. The design automatically converged at nine iterations with horizontal displacements at joints 1 and 2 equal to their limiting values. The final design is completely displacement dominated.

This problem was also solved with six load conditions and 25 independent members with insignificant changes in the final design achieved and the number of analyses required to achieve it. The 2-load condition, 8-independent-member problem just discussed is derived from the 6-load condition, 25-independent-member problem by a priori application of symmetry requirements.

E. Seventy-Two-Member Space Truss

This structure, shown in Fig. 5, has been studied previously in Refs. 1 and 2, and 4-6. All members are aluminum with $E = 10 \times 10^6$ psi and $\rho = 0.1$ lb/in.^{3}. Stress limits of $\pm 25{,}000$ psi are imposed on all members. Displacement limits of ± 0.25 in. in the x and y directions are imposed on the four top nodes. A lower limit of 0.1 in.^{2} is imposed on all members. Design-variable linking is used. Members are placed in 16 groups as in Table 36 of Ref. 2. Thus, there are 16 independent design variables, and two load conditions are considered. These are the same as those of Table 38 of Ref. 2. Table 6 gives the final designs obtained for two initial values of η, and compares these with previous results. The design procedure was started with all members equal to 100 in.^{2}. Starting with $\eta = 0.15$, it was noted that at iteration 8, a weight of 394 lb was achieved, but the procedure continued until iteration 10 when it was automatically stopped with a weight of 388 lb. At the optimum, in the second load condition, the first four members had their stress equal to their limiting values, while the displacements of node 1 in the x and y directions were 2.1% below their specified limits.

F. Two-Hundred-Member Planar Truss

This structure, previously studied in Ref. 13, is given in Fig. 8 of Ref. 13. All members are steel with $E = 30 \times 10^6$ psi and $\rho = 0.283$ lb/in.^{3}. A stress limit of $+10{,}000$ psi is imposed on all members, and displacement limits of ± 0.5 in. are imposed on all nodes in both directions. The structure is symmetric

Table 4 Final design comparison, twenty-two-member space truss

Group no.	Case 1 Sheu and Schmit[12]	Case 1 This paper	Case 2 Sheu and Schmit[12]	Case 2 This paper	Case 3 Sheu and Schmit[12]	Case 3 This paper
1	2.6288	2.5627	2.6101	2.5262	2.5657	2.4902
2	1.1624	1.5530	1.4324	1.9529	1.1331	1.8126
3	0.3433	0.2813	0.587	0.5475	0.0	0.0
4	0.4231	0.5124	0.0	0.0	0.6461	0.6581
5	2.7823	2.6261	2.7861	2.5900	2.6738	2.5442
6	2.1726	2.1314	2.0891	2.2178	2.1768	2.2419
7	1.9523	2.2128	2.0935	2.2630	2.1613	2.2799
Wt, lb	1024.80	1034.74	1028.07	1040.51	1029.35	1040.47
Analyses	...[a]	5	...[a]	6	...[a]	6

[a] Not applicable.

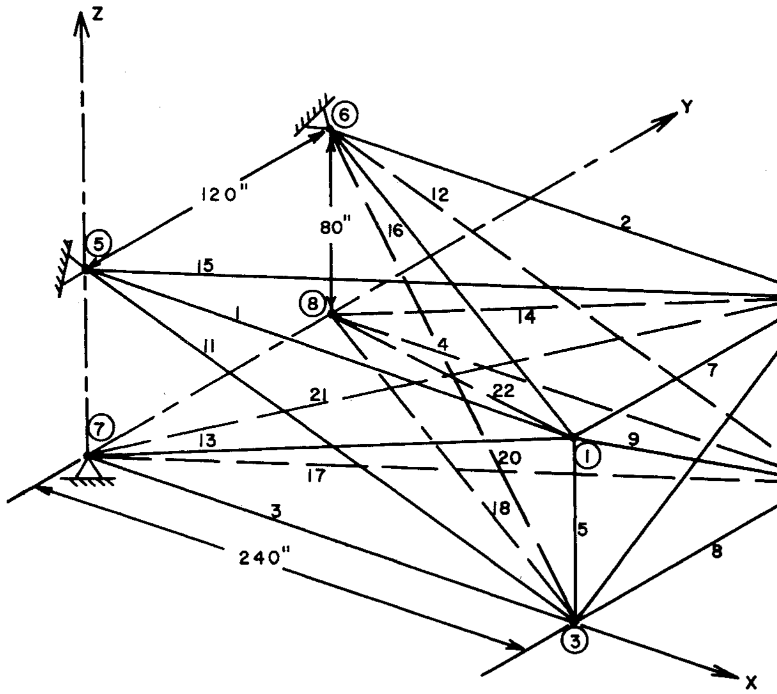

Fig. 3 Twenty-two-member space truss.

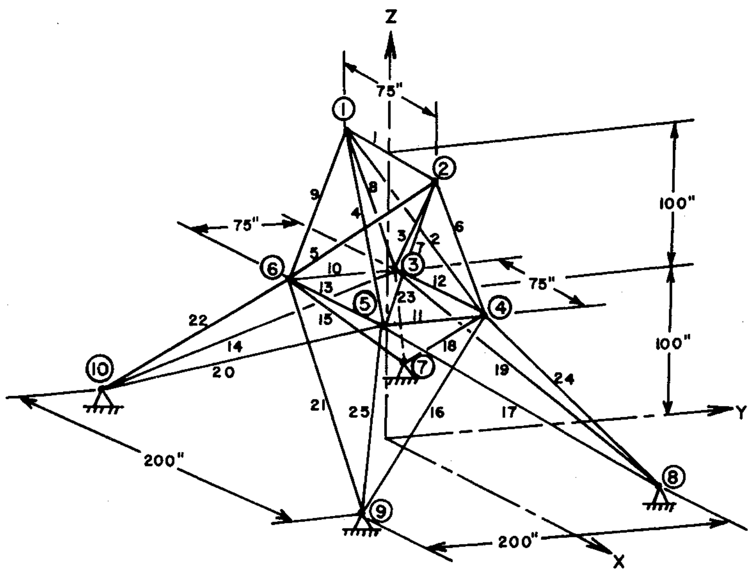

Fig. 4 Twenty-five-bar transmission tower truss.

Table 5 Final designs, twenty-five-member transmission tower truss

Members in group no.	Final cross-sectional areas, in.^{2} Schmit and Miura NEWSUMT[2]	CONMIN[2]	Schmit and Farshi[1]	Venkayya[4]	Gellatly and Berke[5]	Dobbs and Nelson[8]	Rizzi[9]	This paper
1	0.010	0.166	0.010	0.028	0.0100	...[a]	0.01	0.01
2	1.985	2.017	1.964	1.942	2.0069	...	1.9884	1.755
3	2.996	3.026	3.033	3.081	2.9631	...	2.9914	2.869
4	0.010	0.087	0.010	0.010	0.0100	...	0.01	0.01
5	0.010	0.097	0.010	0.010	0.0100	...	0.01	0.01
6	0.684	0.675	0.670	0.693	0.6876	...	0.684	0.845
7	1.677	1.636	1.680	1.678	1.6784	...	1.6767	2.011
8	2.662	2.669	2.670	2.627	2.6638	...	2.6627	2.478
Final wt, lb	545.172	548.475	545.225	545.49	545.36	553.4	545.163	553.94
Analyses needed	10	9	17	7	8	10	10	9

[a] Areas not reported.

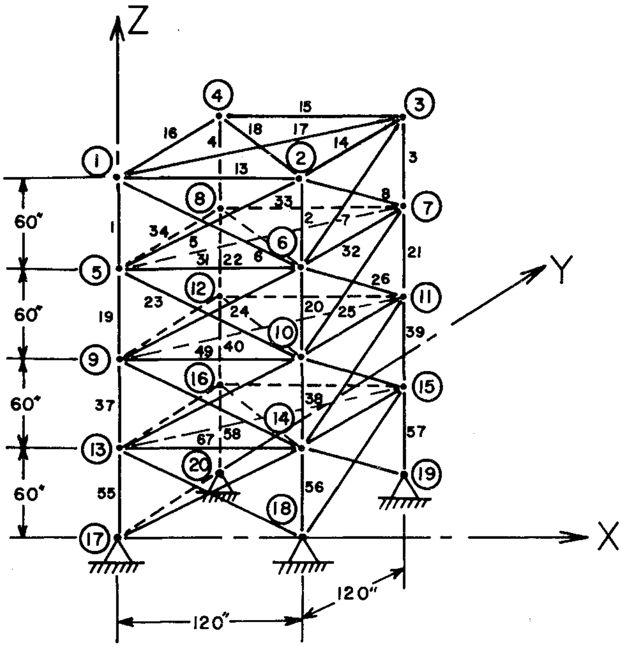

Fig. 5 Seventy-two-member space truss.

about the vertical centerline. This reduces the number of independent design variables to 105. Three load conditions are considered (refer to Fig. 8 of Ref. 13): 1) 1 K in positive x direction at all nodes on the vertical line connecting nodes 1-71; 2) 10 K in negative y direction at all nodes except nodes 76 and 77; and 3) load conditions 1 and 2 acting together. The final design obtained is given in Table 12 of Ref. 15. The final weight of 32,996 lb obtained with eight analyses and 34.35 min of CPU time on an IBM 360/65 compares favorably with the weight of 31,020 lb obtained in Ref. 13 in 90 min of CPU time on an IBM-7094-II-7044-DCS, and comparing the design obtained by the present method, as documented in Ref. 15, with that obtained by Ref. 13 indicates that they are somewhat different. Results of several solutions obtained by the method of this paper indicate that the region of the optimum is flat; i.e., designs of significantly varying member sizes are possible for essentially the same weight.

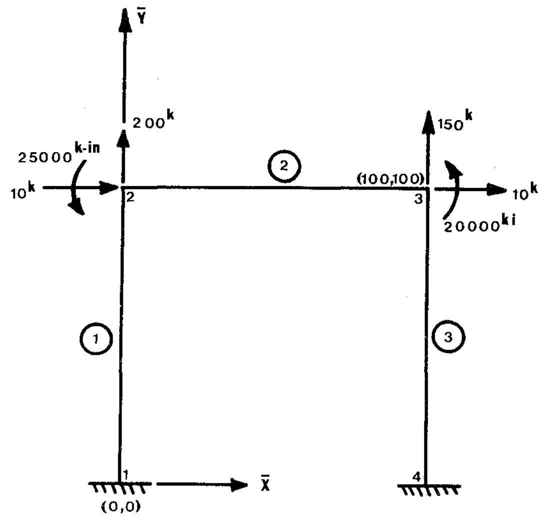

Fig. 6 Three-member frame.

The 32,996-lb design obtained here and the 31,020-lb design of Ref. 13 both have one displacement constraint active at the optimum. This is the displacement of node 5 in the y direction. Very recently, Ref. 16 has reported a weight of 28,924 lb for this structure.

G. Three-Member Frame

The structure is shown in Fig. 6. It is a three-member rigid frame. Each member is treated as one finite element. Axial, shear, and bending moment forces are included in the formulation, resulting in 6 degrees of freedom per element and 3 degrees of freedom per joint. The material is steel with $E=30\times10^6$ psi and $\rho=0.283$ lb/in.^{3}. The design variable for each member is the cross-sectional area A. The section modulus S and moment of inertia I are related to area as $S=9A$ and $I=75A$. These relationships were chosen to give sections representative of available wide flange shapes while maintaining the linearity among A, S, and I. The stress limits for all members are $\pm24{,}000$ psi. One load condition, as shown in Fig. 6, is imposed. Three cases are considered. Cases 1 and 2 include the preceding stress limits and the following displacement constraints; case 1 has the displacements of

Table 6 Final designs, seventy-two-member truss

	Final cross-sectional areas, in.^{2}							
Members of group	Schmit and Miura NEWSUMT[2]	Schmit and Miura CONMIN[2]	Schmit and Farshi[1]	Venkayya[4]	Gellatly and Berke[5]	Berke and Khot[6]	This paper ($\eta=0.1$)	This paper ($\eta=0.15$)
1	0.1565	0.1558	0.1585	0.161	0.1492	0.1571	0.1494	0.1519
2	0.5458	0.5484	0.5936	0.557	0.7733	0.5385	0.5698	0.5614
3	0.4105	0.4105	0.3414	0.377	0.4534	0.4156	0.4434	0.4378
4	0.5699	0.5614	0.6076	0.506	0.3417	0.5510	0.5192	0.5317
5	0.5233	0.5228	0.2643	0.611	0.5521	0.5082	0.6234	0.5814
6	0.5173	0.5161	0.5480	0.532	0.6084	0.5196	0.5231	0.5273
7	0.1000	0.1000	0.1000	0.100	0.1000	0.1000	0.100	0.100
8	0.1000	0.1133	0.1509	0.100	0.1000	0.1000	0.1963	0.1583
9	1.267	1.268	1.1067	1.246	1.0235	1.2793	1.2076	1.2526
10	0.5118	0.5111	0.5792	0.524	0.5421	0.5149	0.5208	0.5244
11	0.1000	0.1000	0.1000	0.100	0.1000	0.1000	0.100	0.100
12	0.1000	0.1000	0.1000	0.100	0.1000	0.1000	0.100	0.100
13	1.885	1.885	2.0784	1.818	1.4636	1.8931	1.7927	1.8589
14	0.5125	0.5118	0.5034	0.524	0.5207	0.5171	0.5223	0.5259
15	0.1000	0.1000	0.1000	0.100	0.1000	0.1000	0.100	0.100
16	0.1000	0.1000	0.1000	0.100	0.1000	0.1000	0.100	0.100
Final wt, lb	379.640	379.792	388.63	381.2	395.97	379.67	386.718	387.67
Analyses	9	8	22	12	9	5	13	10

Table 7 Final design comparison for three-member frame

Member no.	Case 1 Briggs[14]	Case 1 SUMT	Case 1 This paper ($\eta = 0.15$)	Case 2 SUMT	Case 2 This paper ($\eta = 0.15$)	Case 3 SUMT	Case 3 This paper ($\eta = 0.15$)
1	19.74	19.68	19.81	18.34	17.78	6.22	6.43
2	105.38	105.43	105.39	134.0	130.07	47.74	46.42
3	30.13	30.12	30.18	64.44	69.31	21.87	23.04
Vol. in.^{3}	15,525	15,526	15,538	21,677	21,716	7584	7589
Analyses	...[a]	...[a]	6	...[a]	9	...[a]	7
CPU, s[b]	10.19	42.0	1.17	44.09	1.62	68.9	1.27

[a] Not applicable. [b] All times on IBM 360/65.

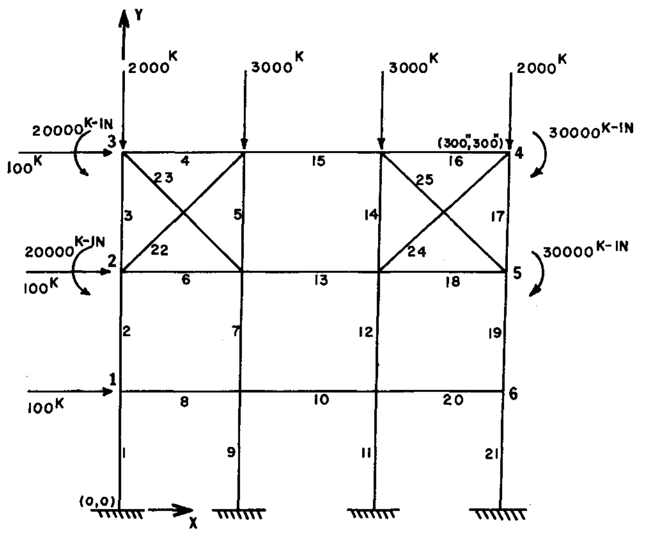

Fig. 7 Twenty-five-member frame.

joints 2 and 3 limited to ± 0.2 in. in the x and y directions and case 2 has the same displacements limited to ± 0.07 in. For case 3, the stress limits are ignored and only displacement constraints of ± 0.2 in. at joints 2 and 3 in both directions are considered. Table 7 gives the results of these three cases and compares them to previously obtained results. It can be seen that excellent agreement has been obtained at a fraction of the CPU time required for these previous results.

Initial designs for Briggs[14] and SUMT were uniform at 75 in.^{2}, and those for the method of this paper uniform at 100 in.^{2}. The η values of Table 7 were constant during the design process.

H. Twenty-Five-Member Frame

The structure is shown in Fig. 7. Members are defined as in Sec. IV. G. One load condition is considered as shown in Fig. 7, and one finite element is used per member. All members are 100 in. in length, except the diagonal members which are 141.4 in. long. Stress limits are $\pm 24{,}000$ psi for all members. Two cases are considered: case 1 has the preceding stress limits and displacement limits of ± 3.0 in. at joints 1-6 in both directions; case 2 has the preceding stress limits and displacement limits of ± 0.05 in. at joints 1-6 in both directions. The minimum member size is 5 in.^{2}. Results are shown in Table 8. Both cases were started with all members equal to 100 in.^{2}. Case 1 is compared with results from Ref. 14 with excellent agreement in designs. The method of this paper can be seen to produce the optimal design with a drastic reduction in the CPU time required for the method of Ref. 14. The design for case 1 is fully stressed at the optimum and the displacement limits are inactive. The case 2 design is displacement constrained with no active stress constraints. No previous results were available for comparison. The η values given in Table 8 did not change during the design process.

Table 8 Final design comparison for twenty-five-member frame

Member no.	Member cross-sectional areas, in.^{2} Case 1 Briggs[14]	Case 1 This paper ($\eta = 0.1$)	Case 2 This paper ($\eta = 0.1$)
1	138.00	129.55	337.79
2	148.58	153.26	293.07
3	154.08	151.29	162.15
4	28.34	31.63	69.68
5	128.93	133.64	170.02
6	5.00	5.00	52.19
7	130.10	131.58	217.43
8	15.72	23.37	108.73
9	162.97	170.80	233.83
10	5.00	5.00	110.55
11	120.97	119.91	170.29
12	111.06	110.78	181.92
13	5.00	5.00	87.35
14	122.00	123.06	109.72
15	5.00	5.00	105.77
16	52.96	54.00	147.31
17	191.76	190.80	233.56
18	5.00	5.00	191.63
19	119.70	123.13	336.97
20	5.00	5.00	199.56
21	123.74	119.32	465.20
22	8.61	5.00	191.92
23	5.00	5.00	88.26
24	5.00	5.00	84.14
25	48.78	48.67	95.59
Vol, in.^{3}	187,421	188,215	463,523
Analyses	...[a]	15	10
CPU, s[b]	1849.00	69.02	38.79

[a] Not applicable. [b] All times on IBM 360/65.

V. Conclusions

A new design algorithm has been developed for stress and displacement constrained trusses and frames under multiple loadings. By means of an extensive set of test problems, the method has been shown to be both accurate and efficient. In all problems studied, known results were reproduced very closely, with the number of structural analyses required in the iterative process approximately the same as the number required by the current most efficient methods. When it is considered that the computational effort required per iteration for the method of this paper is considerably less than that required for all other current methods, and that the core storage required is essentially only that required for the analysis capability, the present method can be seen to be very simple as well as highly efficient.

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